MATHEMATICS

## Linear ODE of First Order

$$\frac{dy}{dx} + a_n(x)y = h(x) \Rightarrow y(x) = \frac{1}{P}\int P h\, dx + \frac{c}{P}, \quad P = e^{\int a_n(x)dx}$$

## Linear Equations with Constant Coefficients

$y = e^{rx} \Rightarrow$ $r \to n$ different roots $\Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$

$r \to$ repeated roots $\Rightarrow c_1 e^{r_1 x} + c_2 x e^{r_2 x} + c_3 x^2 e^{r_3 x} + \dots \quad r_1 = r_2 = r_3 = \dots$

$r \to$ imaginary roots $\Rightarrow y = e^{ax}(c_1 \cos bx + c_2 \sin bx)$

## Particular Solution for Constant Coefficients

| $h(x)$ | Family |
|---|---|
| $x^m$ | $x^m, x^{m-1}, \dots, x^2, x, 1$ |
| $\sin qx$ | $\sin qx, \cos qx$ |
| $\cos qx$ | $\sin qx, \cos qx$ |
| $e^x$ | $e^x$ |

Homojen kısım aynen bulunur
Particular kısım için $y_p$ equationa yazılır
$y_p$ nin katsayıları bulunur.

## ! Equidimensional Linear D.E. (Cauchy Euler eq)

$$x^n \frac{d^n y}{dx^n} + b_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + b_{n-1} x \frac{dy}{dx} + b_n y = h(x)$$

$x = e^z$, $z = \ln x \Rightarrow dx = e^z dz$

$$\frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz}, \quad x\frac{d}{dx} = \frac{d}{dz}, \quad \frac{d^2}{dx^2} = \frac{1}{e^{2z}}\left(\frac{d^2}{dz^2} - \frac{d}{dz}\right), \quad x^2\frac{d^2}{dz^2} = \frac{d}{dz}\left(\frac{d}{dz}-1\right), \quad \frac{x^m}{dz^m} = \frac{d}{dz}\left(\frac{d}{dz}-1\right)\dots\left(\frac{d}{dz}-n\right)$$

If the equation is homogenous $\Rightarrow y = x^r \Rightarrow$ ch eq $\Rightarrow r(r-1)\dots(r-n+1) + b_1 r(r-1)\dots(r-n+2) + \dots + b_{n-1} r + b_n = 0$

distinct roots $\Rightarrow y = \sum_{k=1}^{n} c_k x^{r_k}$, double root $\Rightarrow y = (c_1 + c_2 \log x) + \sum_{k=3}^{n} c_k x^{r_k}$

## Particular Solution by Variation of Parameters.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = h(x)$$

$y_h = c_1 u_1(x) + c_2 u_2(x)$

$y_p = c_1(x) u_1(x) + c_2(x) u_2(x)$

$c_1'(x) u_1(x) + c_2'(x) u_2(x) = 0$

$c_1'(x) u_1'(x) + c_2'(x) u_2'(x) = \frac{h(x)}{a_0(x)}$

$$c_1'(x) = \frac{\frac{h(x)}{a_0(x)} \cdot u_2(x)}{u_1(x)u_2'(x) + u_1'(x)u_2(x)} \qquad c_2'(x) = \frac{u_1(x)\frac{h(x)}{a_0(x)}}{u_1(x)u_2'(x) + u_1'(x)u_2(x)}$$

## Reduction of Order: $y'' + a_1(x)y' + a_2(x)y = h(x)$

If $u_1(x)$ is known $y = v(x)u_1(x)$.
subst in eq $\Rightarrow v' = w$.

## Series Solutions of Differential eqns.

The Power Series is defined as the limit $\lim_{N\to\infty}\sum_{n=0}^{N} A_n (x-x_0)^n$ provided the limit exist. For these values of $x$ for which the limit exist, the series converge. $\rho = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-x_0| = L|x-x_0|$ The series converges when $|x-x_0| < \frac{1}{L}$ and diverges $|x-x_0| > \frac{1}{L}$. Thus when $L$ exists and finite and interval of convergence will be $\left(x_0 - \frac{1}{L}, x_0 + \frac{1}{L}\right)$ $\frac{1}{L}$: Radius of Convergence

## Power series in DEs

$$y=\sum_{n=0}^{\infty}A_n(x-x_0)^n \quad , \quad y'=\sum_{n=0}^{\infty}nA_n(x-x_0)^{n-1} \quad , \quad y''=\sum_{n=0}^{\infty}n(n-1)A_n(x-x_0)^{n-2}$$

## Singular Points of Second Order Linear DEs

$$\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\,y=0 \quad \left.\begin{matrix} a_1(x) \\ a_2(x) \end{matrix}\right\} \text{ Ordinary or Regular Singular.}$$

Ordinary $\Rightarrow$ $y=\sum_{n=0}^{\infty}A_n(x-x_0)^n$ , regular singular $\Rightarrow$ $y=\sum_{n=0}^{\infty}A_n(x-x_0)^{s+n}$

## The Method of Frobenius

$$\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)\,y=0$$

$\leftarrow$ If Both the coefficient functions are regular at $x_0$ ordinary point

$\downarrow$ Power series $y(x)=\sum_{n=0}^{\infty}A_n(x-x_0)^n$

$\rightarrow$ Otherwise $x_0$ is Singular Point. Check singularity: $(x-x_0)\,a_1(x)$ and $(x-x_0)^2 a_2(x)$ Are both regular at $x_0$

Regular Singular points: Use method of Frobenius $y(x)=\sum_{n=0}^{\infty}A_n(x-x_0)^{s+n}$

$\rightarrow$ $A_0\neq 0$

- Indicial Eqn. $f(s)=0$
- Recurrence Relation Coefficients $A_n$

Indicial Equation $f(s)$

- Two Roots Do not Differ by an integer $\rightarrow$ Two Linearly independent Frobenius solutions
- Repeated Root Necessarily Real $\rightarrow$ Second Frobenius Linearly Independent Solution

$$y_2(x)=C\,U(x)\ln x+x^{s_2}\sum_{n=0}^{\infty}B_nx^n$$

$$y_1(x)=A_0\,u(x)$$

$$B_n=\left\{\frac{d}{ds}\left[A_n(s)\right]\right\}_{s=s_2}$$

- Two Roots Differ by a Positive Integer $\rightarrow$ First Frobenius $y_1(x)$ using the Larger root $s_1$ $\rightarrow$ Second Frobenius Linearly Independent Solution.

$$y_2(x)=C\,u(x)\ln x+x^{s_2}\sum_{n=0}^{\infty}B_nx^n$$

$$y_1(x)=A_0\,U(x)$$

$$B_n=\left\{\frac{d}{ds}\left[(s-s_2)A_n(s)\right]\right\}_{s=s_2}$$

## Fourier Series

Assume $f(x)$ → single valued and periodic with a period of $2L$ in the interval $(a, a+2L)$

If $f(x)$ → finite

and $f(x)$ $f'(x)$ → atleast piecewise continuous in $(a, a+2L)$

Then $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$ , $a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}dx$ , $b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}dx$

This is called as complete fourier series.

For an even funct. fourier Cosine series is valid in $(-L<x<L)$

$f(x) = \frac{A_0}{2} + \sum A_n \cos\frac{n\pi x}{L}$ $A_0 = \frac{2}{L}\int_0^L f(x)\,dx$ $A_n = \frac{2}{L}\int_0^L \cos\frac{n\pi x}{L}dx$

For an odd funct. fourier sine series is valid in $(-L<x<L)$

$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}$

## Exponential form of a Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \quad \left\} \begin{aligned} \cos\frac{n\pi x}{L} &= \frac{e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}}{2} \\ \sin\frac{n\pi x}{L} &= \frac{e^{\frac{in\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i}\end{aligned}\right.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{\frac{in\pi x}{L}} + \left(\frac{a_n+ib_n}{2}\right)e^{-\frac{in\pi x}{L}}\right]$$

$$f(x) = C_0 + \sum_{n=1}^{\infty}\left(C_n e^{in\pi x/L} + C_{-n} e^{-in\pi x/L}\right.$$

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} C_n e^{in\pi x/L} \qquad a<x<a+2L$$

$$C_n = \frac{1}{2L}\int_0^{a+2L} f(x)e^{-n\pi x/L}dx \qquad n = (-\infty, \infty)$$

## Double Trigonometric Series

$$f(x,y) = \sum_{n=0}^{\infty} a_n(y)\cos nx + b_n(y)\sin nx \qquad \text{in } (-\pi, \pi)$$

where $a_n(y) = \sum_{m=0}^{\infty} \alpha_{nm}\cos my + \beta_{nm}\sin my$ , $b_n(y) = \sum_{m=0}^{\infty}\gamma_{nm}\cos my + \eta_{nm}\sin my$.

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\alpha_{nm}\cos nx\cos my + \beta_{nm}\sin my\cos nx + \gamma_{nm}\cos my\sin nx + \eta_{nm}\sin my\sin nx\right]$$

$$\alpha_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx\cos my\,dx\,dy$$

$$\beta_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx\sin my\,dx\,dy$$

$$\gamma_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx\cos my\,dx\,dy$$

$$\eta_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx\sin my\,dx\,dy$$

## Boundary Conditions

1- Dirichlet condition: The dependent variable (func. itself) is specified at each point of a boundary.

* Laplace Eqn: $\nabla^2 U = 0$ together with D.Cs constitute the dirichlet problem.

2- Neumann Condition: Values of the normal derivative $\frac{du}{dn}$ of the unknown function are described on the boundary.

3- Mixed Condition: A linear combination of $u$ and $\frac{du}{dn}$ is prescribed

$$L \frac{\partial U}{\partial n} + hU = f(x,y)$$

4- Cauchy Condition: If one of the independent variables is "t" the values of both the unknown function and its partial derivative w.r.t. "t" are given on the boundary at $t=0$

$U$, $\frac{\partial U}{\partial t}$ are given at $t=0$

## PDEs

1- WAWE EQN. $\nabla^2 U = \frac{1}{a^2} \frac{\partial^2 U}{\partial t^2}$

2- HEAT EQN. $\nabla^2 U = \frac{1}{k} \frac{\partial U}{\partial t}$

3- LAPLACE EQN. $\nabla^2 U = 0$

4- POISSON'S EQN. $\nabla^2 U = f$

5- BEAM EQN. $\frac{\partial^4 U}{\partial x^4} + \frac{1}{U^2} \frac{\partial^2 U}{\partial t^2} = 0$

## LAPLACE EQN $\nabla^2 U$

Cylindrical Coordinates: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{\rho} \frac{\partial}{\partial r} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Spherical Coordinates: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 r}{\partial \theta^2} + \frac{1}{r^2 s^2} \frac{\partial^2}{\partial \phi^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r \tan\theta} \frac{\partial}{\partial \theta}$